Cointegration for Seasonal Time Series Processes

by

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ABSTRACT

This paper examines the types of cointegration which can apply between seasonal time series processes. In particular, three types are identified, which are referred to as seasonal cointegration, periodic cointegration and nonperiodic cointegration. The circumstances in which each of these types of cointegration is relevant is examined in a bivariate context and it is shown that, even when both series are first order nonstationary, the possible form(s) for any cointegration depends crucially on the univariate unit root properties of the series. When both processes are (conventionally) integrated or periodically integrated, cointegration can only be nonperiodic or periodic and in the latter case the cointegration coefficients are related to the periodic integration coefficients of the separate variables. The richest set of cointegration possibilities emerge when both series are seasonally integrated, since all three types of cointegration can then apply and, indeed, partial cointegration can hold whereby the stationary linear combinations do not apply to all univariate unit roots. Periodic and seasonal cointegration are shown to imply distinct restrictions, while nonperiodic integration is shown to be a special case of seasonal integration. In the light of the analysis here, some comments are made on previously published studies.

1. INTRODUCTION

Cointegration for seasonal time series processes remains a somewhat perplexing issue. It has long been recognised that the seasonal pattern in a univariate time series may be nonstationary, so that seasonal differencing can be required to render such a stationary. Although procedures are now available for testing for unit roots in seasonal time series, the associated issue of how cointegration should be approached remains far from clearcut. On the one hand, Hylleberg *et al.* (1990) view seasonal cointegration in terms of the separate (zero and seasonal frequency) unit roots implied by seasonal differencing. This route has been followed in a number of subsequent analyses, including Lee (1992), Engle *et al.* (1993), Johansen and Schaumberg (1999). On the other hand, cointegration may be considered season by season, with this route leading to what is known as periodic cointegration. Periodic cointegration appears to have been applied first by Birchenhall *et al.* (1989), with subsequent work by Franses and his co-authors, including the theoretical analysis of Boswijk and Franses (1995). The difficulty is that these are usually seen as two distinct routes for cointegration, with the practitioner selecting the one s/he fancies based on prior prejudices and not on a systematic analysis of the data.

Franses (1993, 1995, 1996) has made a good start in comparing the two approaches, but his analysis is incomplete in a number of respects. The intention of the current paper is to provide a more complete analysis of the issue, detailing the circumstances in which seasonal cointegration, periodic cointegration and the usual constant parameter (nonseasonal) cointegration approaches are applicable. We also consider a relatively parsiminous nested approach which allows the possibilities to be distinguished.

The periodic representation of a time series is fundamental to our approach (and, indeed, that of Franses). In this representation, the quarterly observations for the variable x (say) can be denoted as $x_{s\tau}$, where s relates to the season ($s = 1, 2, 3, 4$) and τ to the year ($\tau = 1, ..., T_\tau$). Conventionally, the first available observation is assumed to relate to quarter 1, and hence is denoted x_{11}. The 4×1 annual vector of observations for year τ can be written $X_\tau = (x_{1\tau}, x_{2\tau}, x_{3\tau}, x_{4\tau})'$. For simplicity, throughout most of this paper we assume quarterly time series. Also to keep the arguments as simple as possible, we initially investigate cointegration issues assuming two variables, namely y and x.

Prior to the substantial discussion of cointegration, Section 2 clarifies univariate concepts in this seasonal context and also presents definitions of cointegration used later. Section 3 then details the implications of the various types cointegration and lays out the relationships between them. A concluding section provides some discussion.

2. PRELIMINARY DEFINITIONS

2.1. Univariate Concepts

The unit root properties of the univariate variable $x_{s\tau}$ can be represented as cointegrating relationships between the elements of the vector X_τ; see Osborn (1993) and, for greater detail, Franses (1994). Thus, we can write X_τ in the VAR($P+1$) representation

$$\Delta_4 X_\tau = \Pi X_{\tau-1} + \sum_{j=1}^{P} \Phi_j \Delta_4 X_{\tau-j} + U_\tau \tag{2.1}$$

where $\Delta_4 = 1 - L^4$ is the annual difference operator, so $\Delta_4 x_{s\tau} = x_{s\tau} - x_{s,\tau-1}$, while $U_\tau \sim iid(0, \Sigma)$ with Σ being positive definite. Indeed, we will use the notation U_τ to refer to a generic white noise vector with these properties. We also use the lag operator L and it is important to appreciate that this operates on the season and not the year, so that $Lx_{s\tau} = x_{s-1,\tau}$ with $L^4 x_{s\tau} = x_{s\tau} - x_{s,\tau-1}$. It should also be noted that when the lagged variable crosses a year, such that $L^i x_{s\tau} = x_{s-i,\tau}$ with $s - i \leq 0$, then it is understood that $x_{s-i,\tau} \equiv x_{s-i+4,\tau-1}$.

Our first definition clarifies what we mean by first order nonstationarity in the present context.

> *Definition 1.* The quarterly time seies process x is *first order nonstationary* when each of the four annual processes for the quarters, namely $x_{s\tau}$ ($s = 1, 2, 3, 4$), is itself integrated of order 1.

Thus, if x is first order nonstationary, then each of the four separate processes $x_{s\tau}$ ($s = 1, 2, 3, 4$) is a conventional $I(1)$ process when considered over years $\tau = 1, 2, \ldots$. In particular, this definition rules out cases where some individual $x_{s\tau}$ ($s = 1, 2, 3, 4$) are stationary and others integrated, since this seem to be implausible in the context of the overall series $x_{s\tau}$ ($s = 1, 2, 3, 4, \tau = 1, 2, \ldots$).

Important special cases of interest to us, discussed in the time series literature to date, are set out in Definition 2.

> *Definition 2.* For the quarterly time series process x, specific forms of first order nonstationarity include:
>
> (i) x is *(nonseasonally and nonperiodically) integrated*, or $x \sim I(1)$, when Π has rank 3 and the three cointegrating relationships can be written as the quarterly differences $x_{2\tau} - x_{1\tau}$, $x_{3\tau} - x_{2\tau}$, $x_{4\tau} - x_{3\tau}$;
>
> (ii) x is *periodically integrated*, or $y \sim PI(1)$, when Π has rank 3 and the three cointegrating relationships can be written as $x_{2\tau} - \alpha_2 x_{1\tau}$, $x_{3\tau} - \alpha_3 x_{2\tau}$, $x_{4\tau} - \alpha_4 x_{3\tau}$ with at least one $\alpha_s \neq 1$ ($s = 2, 3, 4$);
>
> (iii) x is *seasonally integrated*, denoted $x \sim SI(1)$, when Π has rank zero and hence $\Pi = 0$.

Notice that the appropriate transformation to remove the nonstationarity in x differs across the three cases. Although each implies the separate series $\Delta_4 x_{s\tau}$ are stationary for $s = 1,2,3,4$, such annual differencing represents over-differencing when the vector X_τ is considered, except when $x \sim SI(1)$. In other words, due to the presence of cointegrating relationships in the remaining two cases above, common trends are present between the annual series $x_{s\tau} (s = 1,2,3,4)$ and the multivariate Wold representation

$$\Delta_4 X_\tau = C(L^4)\varepsilon_\tau$$

(where $\varepsilon_\tau \sim iid$ has mean vector 0 and positive definite covariance matrix) implies a noninvertible moving average for $\Delta_4 X_\tau$. This noninvertibility is a direct consequence of the reduced rank of $C(1)$ established in Granger Representation Theorem (Engle and Granger, 1987).

It should be noted that when $x \sim I(1)$, then the three cointegrating relationships specified above also imply that the remaining first difference $x_{1\tau} - x_{4,\tau-1}$ is stationary[1]. Similarly, when $x \sim PI(1)$ the three cointegrating relationships noted above imply that $y_{1\tau} - \alpha_1 y_{4\tau}$ is stationary, with $\alpha_1 = 1/\alpha_2\alpha_3\alpha_4$. In practice, periodic integration usually has $\alpha_s > 0$ for $s = 1,2,3,4$, but this nonnegativity is not strictly required.

The case of seasonal integration is an extreme one in the sense that it implies no cointegration between the series for the four quarters of the year. In other words, as emphasised by Osborn (1993), it implies that four distinct unit root processes drive the observed values for the four quarters. Based on the seasonal unit root analysis of Hylleberg, Engle, Granger and Yoo (1990), or HEGY, Franses (1994) sets out other possibilities where Π in (2.1) contains between 1 and 3 specified cointegrating relationships. The HEGY analysis is based on the factorisation

$$\begin{aligned} \Delta_4 &= (1-L)(1+L+L^2+L^3) \\ &= (1-L)(1+L)(1 \pm iL) \end{aligned} \tag{2.2}$$

where $i = \sqrt{-1}$. All factors in (2.2) are nonstationary, with the second line showing clearly that the four factors are each of modulus one.

Another point will be important in the later discussion of cointegration. That is, the HEGY analysis is based on the vector of variables

$$\begin{bmatrix} x_{s\tau}^{(1)} \\ x_{s\tau}^{(2)} \\ x_{s\tau}^{(3)} \\ x_{s\tau}^{(4)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{s\tau} \\ x_{s-1,\tau} \\ x_{s-2,\tau} \\ x_{s-3,\tau} \end{bmatrix} \tag{2.3}$$

where, for later use, the nonsingular transformation matrix used here is defined as T,

[1] By summing the three cointegrating relationships specified above, it is clear that $y_{1\tau} - y_{4\tau}$ must be stationary. However, since $y_{4\tau} - y_{4,\tau-1}$ is stationary, it follows that $y_{1\tau} - y_{4,\tau-1}$ is also stationary.

so

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}. \tag{2.4}$$

The use of the HEGY variables defined by (2.3) rather than the observed quarterly series is important for interpretation. When $x \sim SI(1)$ then, by construction, the process $x_{s\tau}^{(1)}$ has the single unit root $+1$, $x_{s\tau}^{(2)}$ has the single unit root -1, while $x_{s\tau}^{(3)}$ and $x_{s\tau}^{(4)}$ contain only the complex pair of roots $\pm i$. Thus, HEGY associate the nonstationarity in x directly with the zero frequency unit root $+1$ and the seasonal unit roots $-1, \pm i$ through the use of these variables. The periodic approach, on the other hand, effectively views the observed series $x_{s\tau}$ $(s = 1, 2, 3, 4)$ as four separate $I(1)$ processes, with one unit root attached to each. The matrix T captures the relationship between the two approaches and is discussed in detail in Ghysels and Osborn (2001). The important point for our discussion is that when $x \sim SI(1)$ four unit roots are present in X_τ and the choice between the HEGY and periodic views of the source of the nonstationarity is purely a matter of interpretation.

Franses (1994) proposes that the Johansen (1988) cointegration methodology should be used to determine the rank of Π and, conditional on this rank, to test the restrictions implied by the possibilities of interest. However, this procedure is highly parameterised in the seasonal context, leading to the possibility that it may be unable to discriminate between the different cases outlined above.

Although very important in practice, it is not the purpose of the present paper to examine the issues concerned with determining the nature of the nonstationarity in univariate X_τ. Rather, the purpose here is to consider cointegration. In such a multivariate context, each quarterly time series process has a VAR representation analogous to (2.1). Where necessary, with two variables y and x we distinguish the matrices which determine the type of nonstationarity in each as Π_y and Π_x respectively.

2.2. Cointegration for Seasonal Processes

In the context of seasonal variables, three types of cointegration have been used to date. We refer to these as seasonal cointegration (following HEGY, Engle *et al.*, 1993, Johansen and Schaumburg, 1999), periodic integration (used by Birchenhall *et al.*, 1989, Boswijk and Franses, 1995, and others) and the conventional form of cointegration, which we term nonperiodic cointegration.

Taking the case of two variables (x and y) for simplicity, we first define seasonal cointegration from the analysis of Johansen and Schaumburg (1999) as follows:

> *Definition 3.* For quarterly time series processes $x, y \sim SI(1)$, *full seasonal cointegration* holds when a cointegrating relationship applies between each of the following pairs of transformed processes:

(i) $(1 + L + L^2 + L^3)y_{s\tau}$ and $(1 + L + L^2 + L^3)x_{s\tau}$;

(ii) $(1 - L + L^2 - L^3)y_{s\tau}$ and $(1 - L + L^2 - L^3)x_{s\tau}$;

(iii) $[L(1 - L^2) + i(1 - L^2)]y_{s\tau}$ and $[L(1 - L^2) + i(1 - L^2)]x_{s\tau}$;

(iv) $[L(1 - L^2) - i(1 - L^2)]y_{s\tau}$ and $[L(1 - L^2) - i(1 - L^2)]x_{s\tau}$.

Further, for such processes *zero frequency cointegration* is said to apply when the variables of (i) cointegrate, *cointegration at the semi-annual frequency* holds when cointegration applies between the variables of (ii), while *cointegration at the annual frequency* holds when the complex pairs in (iii) and (iv) are cointegrated.

Notice that the definition requires both x and y to be seasonally integrated and hence to have all unit roots implied by the factorisation (2.2). The cointegrating relationship for (i) removes the zero frequency unit root (the unit root 1) present in each of y and x; we denote the cointegrating vector as $(1, -k_1)$. Here, and throughout the paper, we normalise the cointegrating vector so that the coefficient on (transformed) y is unity. Thus, the stationary linear combination implied by the cointegration in (i) can be written as

$$y_{s\tau} + y_{s-1,\tau} + y_{s-2,\tau} + y_{s-3,\tau} - k_1(x_{s\tau} + x_{s-1,\tau} + x_{s-2,\tau} + x_{s-3,\tau}). \tag{2.5}$$

The cointegration in (ii) relates to the semi-annual frequency unit root (the unit root -1), and we denote the cointegrating vector as $(1, -k_2)$ with the variable

$$y_{s\tau} - y_{s-1,\tau} + y_{s-2,\tau} - y_{s-3,\tau} - k_2(x_{s\tau} - x_{s-1,\tau} + x_{s-2,\tau} - x_{s-3,\tau}) \tag{2.6}$$

stationary. From (2.3) it is clear that the cointegrating relationships of (2.5) and (2.9) relate to the pairs of variables $(x^{(1)}_{s\tau}, y^{(1)}_{s\tau})$ and $(x^{(2)}_{s\tau}, y^{(2)}_{s\tau})$ respectively.

The analysis in the case of $\pm i$ is less straightforward and deserves a little discussion. Because the variables in (iii) and (iv) form complex conjugate pairs, the two (complex) cointegrating relationships must themselves form a complex pair. Continuing to normalise the coefficient of the transformed y variable to unity, the cointegrating vectors can be represented as $(1, -k_R \pm ik_I)$. It can then be seen that cointegration for these series implies that the complex conjugate pair of time series

$$\begin{aligned} &y_{s-1,\tau} - y_{s-3,\tau} - k_I x_{s\tau} - k_R x_{s-1,\tau} + k_I x_{s-2,\tau} + k_R x_{s-3,\tau} \\ &+i[y_{s\tau} - y_{s-2,\tau} - k_R x_{s\tau} + k_I x_{s-1,\tau} + k_R x_{s-2,\tau} - k_I x_{s-3,\tau}] \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &y_{s-1,\tau} - y_{s-3,\tau} - k_I x_{s\tau} - k_R x_{s-1,\tau} + k_I x_{s-2,\tau} + k_R x_{s-3,\tau} \\ &-i[y_{s\tau} - y_{s-2,\tau} - k_R x_{s\tau} + k_I x_{s-1,\tau} + k_R x_{s-2,\tau} - k_I x_{s-3,\tau}] \end{aligned} \tag{2.8}$$

are stationary. These are, however, inconvenient and not easily interpretable because they involve complex-valued variables and coefficients. This can be avoided, as follows.

Since any linear combination of stationary series is itself stationary, then adding (2.7) and (2.8) implies that

$$y_{s-1,\tau} - y_{s-3,\tau} - k_I x_{s\tau} - k_R x_{s-1,\tau} + k_I x_{s-2,\tau} + k_R x_{s-3,\tau} \tag{2.9}$$

is stationary, while subtracting them implies that

$$y_{s\tau} - y_{s-2,\tau} - k_R x_{s\tau} + k_I x_{s-1,\tau} + k_R x_{s-2,\tau} - k_I x_{s-3,\tau} \tag{2.10}$$

is also stationary. These last two cointegrating relationships are mutually linearly independent and hence can be used as the two cointegrating relationships corresponding to the pair of unit roots $\pm i$. Notice, in particular, that the cointegrating relationships of (2.9) and (2.10) are defined without recourse to complex coefficients. Thus, full seasonal cointegration implies the presence of the four cointegrating relations in (2.5), (2.6), (2.9) and (2.10), which will later be used extensively.

Another concept we need is that of periodic cointegration. A convenient definition is that of Boswijk and Franses (1995), which in a bivariate context can be expressed as follows:

> *Definition 4.* The first order nonstationary quarterly processes y, x are said to be *fully periodically cointegrated* if each pair of annual processes $y_{s\tau}, x_{s\tau}$ is cointegrated with cointegrating vector $(1, -k_s^P)$ but not all $k_s^P = k$ for $s = 1, 2, 3, 4$. The variables y, x are *partially periodically cointegrated* if some but not all the pairs $y_{s\tau}, x_{s\tau}$ cointegrate for $s = 1, 2, 3, 4$.

Notice that our definition implies that, for full periodic cointegration, the same cointegrating vector cannot apply for all of the quarters $s = 1, 2, 3, 4$. This is requirement is not placed by Boswijk and Franses, but we make it in order to distinguish full periodic cointegration from nonperiodic cointegration, with the latter defined as follows.

> *Definition 5.* The first order nonstationary quarterly processes y, x are said to be *nonperiodically cointegrated* if each pair of annual processes $y_{s\tau}, x_{s\tau}$ is cointegrated with identical cointegrating vector $(1, -k)$ for $s = 1, 2, 3, 4$.

It should also be remarked that these last two definitions imply that if some (but not all) pairs $y_{s\tau}, x_{s\tau}$ are cointegrated with common cointegrating vector $(1, -k)$, then the variables are partially periodically cointegrated. Nonperiodic cointegration is reserved for the case where the common cointegrating vector applies to *all* pairs over $s = 1, 2, 3, 4$.

It is worth reflecting that periodic and nonperiodic cointegration as defined here require the cointegrating relationship between the annual series $y_{s\tau}$ and $x_{j\tau}$ to apply contemporaneously with $s = j$. This can be viewed as an unnecessary restriction, which is relaxed in the following definition.

Definition 6. The first order nonstationary quarterly processes y, x are said to be *fully asychronously periodically cointegrated at lag i* if each pair of annual processes $y_{s\tau}, x_{s-i,\tau}$ $(i \neq 0)$ is cointegrated with cointegrating vector $(1, -k_s^P)$ with not all $k_s^P = k$ for $s = 1, 2, 3, 4$. The variables y, x are *asynchronously nonperiodically cointegrated at lag i* if the identical cointegrating relationship $(1, -k)$ applies for all pairs $y_{s\tau}, x_{s-i,\tau}$ $(i \neq 0)$ for $s = 1, 2, 3, 4$.

Clearly the first part of this definition can be extended to cover partial asychronous periodic cointegration at lag i.

3. COINTEGRATION POSSIBILITIES

Analogous to the vector representation of a single series as considered in Section 2.1, and following Franses (1995), we can define the vector representation for the stacked bivariate vector $Z_\tau = (y_{1\tau}, y_{2\tau}, y_{3\tau}, y_{4\tau}, x_{1\tau}, x_{2\tau}, x_{3\tau}, x_{4\tau})'$ as

$$\Delta_4 Z_\tau = A Z_{\tau-1} + \sum_{j=1}^{P} \Phi_j \Delta_4 Z_{\tau-j} + U_\tau. \tag{3.1}$$

Since we are not explicitly interested in the vector autoregressive matrices, no confusion should be caused by using the same general notation Φ_j in both (2.1) and (3.1). The rank of A in (3.1) is clearly of interest to us in the context of cointegration. When each of the quarterly processes y, x is first order nonstationary while A has nonzero rank r, we can write $A = \Gamma K'$ where the columns of the $8 \times r$ matrix K contains the cointegrating relationships and hence the $r \times 1$ vector $K' Z_\tau$ is stationary.

Franses (1995) notes that the sum of the number of unit roots in the separate systems for Y_τ and X_τ is the maximum number of unit roots in the system (3.1). Conversely, in terms of cointegration, the number of linearly independent cointegrating relationships in the bivariate system (3.1) must be at least as great as the sum of the numbers in the separate systems for Y_τ and X_τ, since these latter cointegrating relationships also apply in the bivariate system. This immediately gives rise to our first Proposition.

Proposition 1. At most one linearly independent cointegrating relationship can exist between two processess which are either $I(1)$ or $PI(1)$.

The discussion of subsection 2.1 above showed that when x is $I(1)$ or $PI(1)$ then Π_x contains three cointegrating relationships. Consequently, in the common trends representation of Stock and Watson (1988), the elements of X_τ share precisely one common trend. With two variables y, x which are each $I(1)$ or $PI(1)$, the respective single common trend for each case can be represented as ς_τ^y and ς_τ^x respectively. From the analysis of Stock and Watson, it follows that cointegration is equivalent to a single common trend between ς_τ^y and ς_τ^x, and hence at most one cointegrating relationship.

There are, however, a number of different ways of representing such cointegration. These are drawn out in the following four corollaries to this proposition, which consider $I(1)$ and $PI(1)$ processes. Note that seasonal cointegration is inappropriate for such processes because it is based on finding cointegration corresponding to each of the unit roots $+1, -1, \pm i$ present in the annual difference operator Δ_4. Since $I(1)$ and $PI(1)$ processes do not contain unit roots at the seasonal frequencies, cointegration at such frequencies is not appropriate. The Appendix details the proofs for these and corollaries to later Propositions.

Corollary 1.1. For the quarterly processes $y, x \sim I(1)$,

(i) The variables are either nonperiodically cointegrated or not cointegrated. In particular, such variables cannot be seasonally cointegrated or (fully or partially) periodically cointegrated.

(ii) When the processes are nonperiodically cointegrated, they are also asynchronously nonperiodically cointegrated at all lags $i = 1, 2, 3$.

(iii) When the variables are nonperiodically cointegrated, the matrix A has rank seven, with one representation of the linearly independent cointegrating relationships being given by the rows of

$$K'_{II} = \begin{bmatrix} 1 & 0 & 0 & 0 & -k & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -k & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}. \quad (3.2)$$

Notice that here, and in later representations we exploit the univariate properties of x and the cointegrating relationship between each $y_{s\tau}, x_{s\tau}$ $(s = 1, 2, 3, 4)$, with the properties of y then being implied. A triangular representation of this type is particularly useful when x is exogenous.

Corollary 1.2. For quarterly time series processes $y, x \sim PI(1)$,

(i) The variables may be nonperiodically cointegrated, fully periodically cointegrated or not cointegrated. Such variables cannot be seasonally cointegrated or partially periodically cointegrated.

(ii) If the variables are nonperiodically cointegrated they are also asynchronously nonperiodically cointegrated at lags $i = 1, 2, 3$, while if they are periodically cointegrated they are also asynchronously periodically cointegrated at lags $i = 1, 2, 3$;

(iii) When they are cointegrated, the matrix A has rank seven, with linearly independent cointegrating relationships which can be defined as the rows of

$$K'_{PP} = \begin{bmatrix} 1 & 0 & 0 & 0 & -k_1^P & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -k_2^P & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -k_3^P & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -k_4^P \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\alpha_1 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_3 & 1 & 0 \end{bmatrix}. \tag{3.3}$$

(iv) When y, x are cointegrated, then identical periodic integration coefficients, α_s $(s = 1, 2, 3, 4)$ where $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$, apply for both x and y if and only if y, x are nonperiodically cointegrated.

The first two corollaries specify both variables to be either $I(1)$ or $PI(1)$. However, in the third corollary we allow one variable to be $PI(1)$ and the other to be $I(1)$. Although we arbitrarily assume that it is x which is $I(1)$; corresponding arguments apply if $y \sim I(1)$ and $x \sim PI(1)$.

Corollary 1.3. For the quarterly processes $x \sim I(1)$ and $y \sim PI(1)$,

(i) The variables may be fully periodically cointegrated or not cointegrated. Such variables cannot be seasonally cointegrated, nonperiodically cointegrated, or partially periodically cointegrated.

(ii) When the processes are fully periodically cointegrated, they are also asynchronously fully periodically cointegrated at lags $i = 1, 2, 3$.

(iii) When they are cointegrated, the matrix A has rank seven, with linearly independent cointegrating relationships which can be written as the rows of

$$K'_{IP} = \begin{bmatrix} 1 & 0 & 0 & 0 & -k_1^P & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -k_2^P & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -k_3^P & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -k_4^P \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}. \tag{3.4}$$

(iv) The periodic integration coefficients for y and the periodic cointegration vectors are related through $k_1^P = \beta_1 k_4^P, k_2^P = \beta_1\beta_2 k_4^P, k_3^P = \beta_1\beta_2\beta_3 k_4^P$.

We now turn to the nature of contegration for $SI(1)$ variables.

Proposition 2. At most four linearly independent cointegrating relationships can exist between two $SI(1)$ variables.

The $SI(1)$ property implies that each vector Y_τ, X_τ contains four distinct stochastic trends. Cointegration between y and x implies the existence of one or more linearly independent stationary linear combinations across the elements of Y_τ, X_τ. Since no stationary linear combination within the sets $(y_{1\tau}, y_{2\tau}, y_{3\tau}, y_{4\tau})$ or $(x_{1\tau}, x_{2\tau}, x_{3\tau}, x_{4\tau})$ exists, each stationary linear combination must involve at least one $y_{s\tau}$ and at least one $x_{j\tau}$ $(s, j = 1, 2, 3, 4)$. Consequently, a maximum of four linearly independent stationary combinations can exist. To formally establish this last statement, consider the matrix A in (3.1). It is a standard result of linear algebra that elementary row operations can be used to reduce this matrix to Echelon form, with the rank of the matrix then being the number of nonzero rows in this form (see, for example, Hadley, 1961). The Echelon matrix is upper triangular and, because no stationary linear combinations of the $x_{s\tau}$ exist, the final 4 rows must contain only zeros. Thus, A can be of rank at most four, yielding the result stated.

The first corollary explores some possibilities for the cointegrating relationships when the maximum of four exist. For $SI(1)$ processes, the seasonal and periodic approaches to cointegration do not differ in terms of the number of cointegrating relationships being sought, but rather in the nature of the common stochastic trends assumed to underly such cointegration. As we noted in the discussion of subsection 2.1, seasonal integration is defined in relation to the four unit roots $+1, -1, \pm i$ implied by the annual difference operator Δ_4, with one underlying stochastic trend implicitly related to each of these unit roots. The periodic approach, on the other hand, treats the underlying stochastic trends as being the observed annual series, $x_{s\tau}$ $(s = 1, 2, 3, 4)$ themselves. Each approach then examines cointegration in terms of whether stationary relationships exist across the variables which capture the assumed nature of the stochastic trends.

Corollary 2.1. For the quarterly processes $x, y \sim SI(1)$ assume that the matrix A has rank four. Then the possibilities for cointegration include:

(i) Full seasonal cointegration, with matrix of cointegrating vectors which can be written as

$$K'_{SC} = \begin{bmatrix} 1 & 1 & 1 & 1 & -k_1 & -k_1 & -k_1 & -k_1 \\ -1 & 1 & -1 & 1 & k_2 & -k_2 & k_2 & -k_2 \\ -1 & 0 & 1 & 0 & k_R & k_I & -k_R & -k_I \\ 0 & -1 & 0 & 1 & -k_I & k_R & k_I & -k_R \end{bmatrix}; \quad (3.5)$$

(ii) Full periodic cointegration, where the matrix of cointegrating vectors can be represented as

$$K'_{PC} = \begin{bmatrix} 1 & 0 & 0 & 0 & -k_1^P & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -k_2^P & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -k_3^P & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -k_4^P \end{bmatrix}; \quad (3.6)$$

(iii) Nonperiodic cointegration, where the cointegrating matrix is of the form (3.6) with $k_1^P = k_2^P = k_3^P = k_4^P = k$;

(iv) Full asynchronous periodic cointegration at lags $i = 1, 2$ or 3, with cointegrating matrix for the case $i = 1$ being

$$K'_{PC1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -k_1^P \\ 0 & 1 & 0 & 0 & -k_2^P & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -k_3^P & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -k_4^P & 0 \end{bmatrix}; \quad (3.7)$$

(v) Asynchronous nonperiodic cointegration at lags $i = 1, 2$ or 3, which for lag $i = 1$ has cointegrating matrix (3.7) with $k_1^P = k_2^P = k_3^P = k_4^P = k$.

It should also be mentioned that other possibilities exist for cointegration in the case tackled by Corollary 2.1. For example, Franses (1996, pp.180) considers four distinct cointegrating vectors over $s = 1, 2, 3, 4$ relating to the HEGY variables $x_{s\tau}^{(1)}, y_{s\tau}^{(1)}$ or $x_{s\tau}^{(2)}, y_{s\tau}^{(2)}$. Although these could be added to the list of possibilities, cases of this type mix the periodic and seasonal cointegration approaches, which we prefer to keep separate for expositional clarity.

It is, however, important to establish the cases when the seasonal and periodic approaches lead to distinct cointegration possibilities and when they are equivalent. These are set out in the next corollary.

Corollary 2.2. For the quarterly processes $x, y \sim SI(1)$ assume that the matrix A has rank four. Then:

(i) The sets of cointegrating vectors implied by full seasonal cointegration, full periodic cointegration and full asynchronous periodic cointegration at lags $i = 1, 2, 3$ are distinct in the sense that these sets are linearly independent of each other. Therefore, the existence of cointegration of one of these types rules out the others.

(ii) Nonperiodic cointegration is equivalent to a special case of full seasonal cointegration with $k_1 = k_2 = k_R = k$ and $k_I = 0$.

(iii) Each case of asynchronous nonperiodic cointegration at lags $i = 1, 2, 3$ is equivalent to a special case of full seasonal cointegration. In particular,

$i = 1$ implies $k_1 = -k_2 = -k_I = k$, $k_R = 0$,

$i = 2$ implies $k_1 = k_2 = -k_R = k$, $k_I = 0$;

$i = 3$ implies $k_1 = -k_2 = k_I = k$, $k_R = 0$.

The first part of this corollary is important in that it establishes that full seasonal cointegration, full periodic cointegration and full asynchronous periodic cointegration are, indeed, distinct possibilities when both variables are $SI(1)$. This is established in the Appendix by showing that, with distinct cointegrating coefficients k_s, k_s^P across s, none of the matrices (3.5), (3.6) and (3.7) can be obtained from another by using elementary row operations. Its importance is that it makes clear that a blind adherence to following a seasonal or periodic approach to cointegration could be a severe mistake because the failure to find one type of cointegration does not imply that the other type is not present.

The final two parts then consider cases where the seasonal and periodic approaches in fact lead to equivalent sets of (nonperiodic) cointegrating vectors. It also provides a temporal interpretation for some special cases of seasonal cointegration and avoids some of the interpretation difficulties noted by Johansen and Schaumburg (1999) for seasonal cointegration (and especially for cointegration at the annual frequency).

When $x, y \sim SI(1)$ and the matrix A has rank less than four, there are many possibilities for the type of cointegration, including partial seasonal cointegration and partial periodic cointegration.

So far all cases examined have considered cases where y and x contain the same number of unit roots. Clearly, however, there is interest in cases where this is not so. In particular, one series may be $SI(1)$ and the other $I(1)$ or $PI(1)$. Although we arbitrarily assume that x is either $I(1)$ or $PI(1)$ while $y \sim SI(1)$, the roles of x and y can be reversed without changing the essential argument.

Proposition 3. For the quarterly time series processes x, y with $x \sim I(1)$ or $x \sim PI(1)$ and $y \sim SI(1)$, then A has rank at most 4 and hence there can be at most one linearly independent cointegrating relationship between the separate variables y, x.

This Proposition follows because x has a single common stochastic trend ς^x_τ, while y has four stochastic trends Cointegration can apply between ς^x_τ and any one of the four stochastic trends of y. It is impossible for there to be more than one cointegrating relationship across x and y because this would imply a second cointegrating relationship between ς^x_τ and the elements of Y_τ. If such a second cointegrating relationship were present, then (by substitution) there must be a stationary relationship between the elements of Y_τ, which is ruled out by $y \sim SI(1)$.

The Proposition immediately leads to the corollary below.

Corollary 3.1. If quarterly $x \sim I(1)$ with quarterly $y \sim SI(1)$, possibilities for cointegration between x and y include:

(i) Partial periodic cointegration between $x_{s\tau}$ and $y_{s\tau}$ for a single s ($s = 1, 2, 3$ or 4). This single partial periodic cointegrating relationship implies partial asynchronous periodic cointegration between the annual process $y_{s\tau}$ and $x_{s-i,\tau}$ for lags $i = 1, 2, 3$ with the same cointegrating vector applying in all cases.

(ii) Nonperiodic cointegration between $x_{s\tau}$ and the HEGY variable $y^{(1)}_{s\tau}$.

The possibility in (i) allows x to be cointegrated with (just) one of the separate annual processes $y_{s\tau}$. However, because $x_{j\tau} - x_{j-1,\tau}$ is stationary, the same cointegrating relationship applies also between the specific annual process $y_{s\tau}$ and all $x_{j\tau}$ ($j = 1, 2, 3, 4$). Part (ii) allows the possibility of cointegration with the HEGY variable $y^{(1)}_{s\tau}$. The relevant variable for cointegration with $x_{s\tau}$ in this case is $y^{(1)}_{s\tau}$ since this latter variable is $I(1)$ by construction. The cointegration here is nonperiodic, since both variables $x_{s\tau}$ and $y^{(1)}_{s\tau}$ are conventional $I(1)$ variables and hence the same cointegrating relationship applies for all s. Indeed, since $x_{s\tau}, y^{(1)}_{s\tau} \sim I(1)$, Corollary 1.1 applies.

It might be noted that there are other possibilities for the single cointegrating relationship. Nevertheless, the two specified in the corollary reflect the essence of the periodic and seasonal cointegration approaches. It is straightforward to extend these two possibilities to allow $x \sim PI(1)$.

Corollary 3.2. If quarterly $x \sim PI(1)$ with periodic integration coefficients α_j ($j = 1, 2, 3, 4$), while quarterly $y \sim SI(1)$, possibilities for cointegration between x and y include:

(i) Partial periodic cointegration between $x_{s\tau}$ and $y_{s\tau}$ for a single s ($s = 1, 2, 3$ or 4). This partial periodic cointegrating relationship implies partial asynchronous periodic cointegration between the same annual process $y_{s\tau}$ and $x_{s-i,\tau}$ for lags $i = 1, 2, 3$. If the cointegrating vector is $(1, -k^P)$, then at lags $1,2,3$ the partial asynchronous periodic cointegrating vectors are $(1,-\alpha_s k^P)$, $(1,-\alpha_{s-1}\alpha_s k^P)$, $(1,-\alpha_{s-2}\alpha_{s-1}\alpha_s k^P)$ respectively[2].

(ii) Full periodic cointegration between $x_{s\tau}$ and the HEGY variable $y^{(1)}_{s\tau}$ over $s=1,2,3,4$. If the cointegrating vector for $s=4$ is $(1,-k)$, then the vectors for $s=1,2,3$ are $(1,-\alpha_2\alpha_3\alpha_4 k)$, $(1,-\alpha_3\alpha_4 k)$, $(1,-\alpha_4 k)$ respectively.

For part (ii) here, Corollary 1.3 applies because one variable is $I(1)$ and the other is $PI(1)$. However, here it is x which is $PI(1)$. The two possibilities which are specified in Corollaries 3.1 and 3.2 can be illustrated by the following two matrices of cointegrating vectors for the A of (3.1)

$$K'_{IS} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -k^P \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\alpha_1 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_3 & 1 & 0 \end{bmatrix} \tag{3.8}$$

and

$$K'_{PS} = \begin{bmatrix} 1 & 1 & 1 & 1 & -k & -k & -k & -k \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\alpha_1 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_3 & 1 & 0 \end{bmatrix}. \tag{3.9}$$

In (3.8) and (3.9), the more general case $x \sim PI(1)$ is taken, with the $I(1)$ case being implied by $\alpha_1=\alpha_2=\alpha_3=\alpha_4=1$. The former matrix assumes that the cointegration between $x_{s\tau}$ and $y_{s\tau}$ applies for $s=4$. The other cointegrating relationships specified in these two corollaries can be obtained from those shown in the matrices by the use of elementary row transformations.

4. CHOOSING BETWEEN SEASONAL AND PERIODIC COINTEGRATION

It is clear from the possibilities laid out in the previous subection that the univariate properties of the quarterly variables x, y play a crucial role in determining what type of cointegration should be considered. Indeed, when either variable is known to be $I(1)$ or $PI(1)$, then it follows that at most one linearly independent cointegrating relationship can exist between x and y. In such a case it would seem absurd to consider using the general 8×8 matrix A of the bivariate representation (3.1) in order to investigate the nature of this relationship. The richest set of possibilities apply when both variables are seasonally integrated, since up to four linearly independent cointegrating relationships

[2] Note that $\alpha_i \equiv \alpha_{4+i}$ when $j \leq 0$.

can then exist. Even then, however, it seems advisable to directly investigate the nonperiodic, periodic or seasonal cointegration possibilities rather than to attempt to estimate and test the elements of A.

Franses (1993) proposes that the choice between seasonal and periodic cointegration be made in a bivariate context by testing for cointegration between $y_{s\tau}$ and $x_{s\tau}$ season by season. Thus, he advocates taking the periodic perspective and only considering seasonal cointegration if evidence against periodic (or nonperiodic) cointegration is found. Clearly, this does not treat the two approaches symmetrically. In Franses (1995) he refers to estimating the matrix A using information from the univariate specifications and then testing restrictions implied by the different cointegration possibilities. Although he does not specify details of how this should be achieved in general, clearly this is a more satisfactory approach. However, it is highly parameterised because he appears to recommend the estimation of 8×8 coefficient matrices. Our purpose here is to illustrate how the possibilities for two $SI(1)$ processes may be empirically examined in a parsiminous framework.

The key to our method is to note that the cointegrating matrices in (3.5) to (3.7) are nested in the matrices

$$K_S' = \begin{bmatrix} 1 & 1 & 1 & 1 & -b_{11}^S & -b_{12}^S & -b_{13}^S & -b_{14}^S \\ -1 & 1 & -1 & 1 & -b_{21}^S & -b_{22}^S & -b_{23}^S & -b_{24}^S \\ -1 & 0 & 1 & 0 & -b_{31}^S & -b_{32}^S & -b_{33}^S & -b_{34}^S \\ 0 & -1 & 0 & 1 & -b_{41}^S & -b_{42}^S & -b_{43}^S & -b_{44}^S \end{bmatrix} \tag{4.1}$$

and

$$K_P' = \begin{bmatrix} 1 & 0 & 0 & 0 & -b_{11}^P & -b_{12}^P & -b_{13}^P & -b_{14}^P \\ 0 & 1 & 0 & 0 & -b_{21}^P & -b_{22}^P & -b_{23}^P & -b_{24}^P \\ 0 & 0 & 1 & 0 & -b_{31}^P & -b_{32}^P & -b_{33}^P & -b_{34}^P \\ 0 & 0 & 0 & 1 & -b_{41}^P & -b_{42}^P & -b_{43}^P & -b_{44}^P \end{bmatrix}. \tag{4.2}$$

Indeed, these two last matrices are themselves linked through the one-to-one relationship $K_S' = TK_P'$ where T is defined by (2.4). We discuss these two representations in separate subsections below.

4.1. Testing in the Periodic Framework

The representation (4.2) implies that the cointegrating relationships can be represented as

$$Y_\tau = B_P X_\tau + M_1 + U_{1\tau} \tag{4.3a}$$

where we take M_1 to be an unrestricted (4×1) vector of intercepts and $U_{1\tau}$ is a (4×1) vector of stationary zero-mean disturbances. The (i, j) element of B_P is b_{ij}^P of (4.2). This representation assumes that B_P has full rank, and hence that each $y_{s\tau}$ $(s = 1, 2, 3, 4)$ is cointegrated with at least one of the variables $x_{1\tau}, x_{2\tau}, x_{3\tau}, x_{4\tau}$. This is because first-order nonstationarity for each $y_{s\tau}$ is otherwise incompatible with $U_{1\tau}$ being stationary.

A number of authors, including Phillips (1991) and Stock and Watson (1993), have examined hypothesis testing in the framework of a triangular representation. If (4.3a) is augmented by a block of equations

$$\Delta_4 X_\tau = M_2 + U_{2\tau} \tag{4.4}$$

then the complete system for the vector $(Y_\tau', X_\tau')'$ defines such a triangular system. In our context, M_2 is an unrestricted (4×1) vector of intercepts and $U_{2\tau}$ is a (4×1) vector of stationary zero-mean disturbances. Inclusion of M_2 allows nonzero drift terms to be present in the $I(1)$ processes for $x_{s\tau}$ $(s = 1, 2, 3, 4)$[3]. Note that contemporaneous correlation between $U_{1\tau}$ and $U_{2\tau}$ is permitted, and hence X_τ may be endogenous, while the vector $U_\tau = (U_{1\tau}', U_{2\tau}')'$ may also exhibit stationary autocorrelation.

The key result of these studies is that, provided the possible endogeneity of X_τ and autocorrelation are handled in an appropriate way, then inference can be conducted on the elements of B_P using the standard χ^2 distribution. Clearly, since we need to examine cross-equation restrictions in order to test seasonal cointegration, then the simultaneous estimation of all rows of B_P is vital. Stock and Watson (1993) examine a generalised least squares (GLS) method which can be used for this purpose.

The specific restrictions to be tested for full periodic and full seasonal cointegration are as follows. From a straightforward comparison of (4.2) with (3.7), it is obvious that periodic cointegration implies that B_P is diagonal. Clearly, therefore, testing for periodic cointegration implies testing zero restrictions on the twelve off-diagonal elements of this matrix. A test of nonperiodic cointegration further requires testing of the three restrictions $b_{11}^P = b_{22}^P = b_{33}^P = b_{44}^P$.

From (3.5), full seasonal cointegration implies that

$$TB_P = \begin{bmatrix} k_1 & k_1 & k_1 & k_1 \\ -k_2 & k_2 & -k_2 & k_2 \\ -k_R & -k_I & k_R & k_I \\ k_I & -k_R & -k_I & k_R \end{bmatrix}$$

which in turn implies twelve linear restrictions on the elements of B. Using the form of T given in (2.4), it follows that these can be written as the following twelve linearly independent restrictions:

$$\begin{aligned} k_1 &= b_{11}^P + b_{21}^P + b_{31}^P + b_{31}^P = b_{12}^P + b_{22}^P + b_{32}^P + b_{42}^P \\ &= b_{13}^P + b_{23}^P + b_{33}^P + b_{43}^P = b_{14}^P + b_{24}^P + b_{34}^P + b_{44}^P \end{aligned}$$

[3] In practice, in the context of seasonal integration, M_2 might be restricted to have all elements equal in order to impose a common growth in $x_{s\tau}$ across $s = 1, 2, 3, 4$. However, such a restriction is not considered here.

$$
\begin{aligned}
k_2 &= b^P_{11} - b^P_{21} + b^P_{31} - b^P_{31} = -b^P_{12} + b^P_{22} - b^P_{32} + b^P_{42} \\
&= b^P_{13} - b^P_{23} + b^P_{33} + b^P_{43} = -b^P_{14} + b^P_{24} - b^P_{34} + b^P_{44}
\end{aligned}
$$

$$
k_R = b^P_{11} - b^P_{31} = -b^P_{13} + b^P_{33} = b^P_{22} - b^P_{42} = -b^P_{24} + b^P_{44}
$$

$$
k_I = b^P_{12} - b^P_{32} = -b^P_{14} + b^P_{34} = -b^P_{21} + b^P_{41} = b^P_{23} - b^P_{43}
$$

The three further restrictions $k_1 = k_2 = k_R$ and $k_I = 0$ then give rise to nonperiodic cointegration.

Therefore, the possibilities of full seasonal cointegration, full periodic cointegration and nonperiodic cointegration can be tested through (4.3a). Since nonperiodic cointegration a special case of both full seasonal cointegration and full periodic cointegration (by Corollary 2.2 above), then the two sets of fifteen restrictions implied by nonperiodic cointegration simply represent two parameterisations of the same restrictions. Therefore, in terms of testing these fifteen restrictions, it is irrelevant whether the route taken is through seasonal cointegration or through periodic cointegration.

5. CONCLUDING REMARKS

The results of the Section 3 enable us to comment on various empirical studies of cointegration in the context of seasonal time series processes. It is interesting to note that most applications of both periodic and seasonal cointegration relate to consumption and income data; examples include Birchenhall *et al.* (1989), Hylleberg *et al.* (1990), Engle *et al.* (1993), Franses and Kloek (1995), Franses and Paap (1995), Franses (1996, pp.204-207) and Herwartz (1997). It is notable that even when the papers concerned with periodic cointegration consider the univariate properties of time series, the implications of these for the nature of cointegration is generally ignored.

The pitfalls of failing to recognise the implications of the unit root properties of the individual series for cointegration in a seasonal context may be illustrated by considering the example analysis of concumption and income in Sweden given by Franses (1996, pp.204-207). An initial univariate analysis indicates that the vector, X_τ say, for consumption has one unit root. Thus, $x \sim I(1)$ or $PI(1)$. Income is judged to be $SI(1)$, and hence $y \sim SI(1)$. Therefore, the situation is that analysed in Corollaries 3.1 and 3.2, with only one cointegrating relation possible across $x_{s\tau}, y_{s\tau}$ $(s = 1, 2, 3, 4)$. However, an analysis of the annual series $x_{s\tau}, y_{s\tau}$ for $s = 1, 2, 3, 4$ provides evidence of partial periodic cointegration for both $s = 2$ and $s = 4$. As our analysis in Section 3 shows, such a conclusion is logically impossible.

Our analysis also enables to contribute to an on-going discussion about the nature of cointegration implicity assumed in the seminal paper by Davidson *et al.* (1978). Because they assume $y_{s\tau} - kx_{s\tau}$ is stationary[4], the cointegration is specified by Davidson

[4]More precisely, they assume that $y_{s\tau} - x_{s\tau}$ is stationary. However, most subsequent analyses relax the restriction $k = 1$.

et al. as nonperiodic. Based on their different approaches, Hylleberg *et al.* (1990) and Franses (1996, p.180) generalise the Davidson *et al.* model to consider seasonal and periodic cointegration respectively. However, if the nonperiodic restriction is valid, both approaches should lead to equivalent specifications. It is also worth noting that Harvey and Scott (1994) criticise the use by Davidson *et al.* of the relationship $y_{s\tau} - kx_{s\tau}$, arguing that, in the presence of univariate seasonal unit roots, the cointegration should be (in our notation for the HEGY transformed variables) of the form $y_{s\tau}^{(1)} - kx_{s\tau}^{(1)}$. Indeed, Harvey and Scott state that $y_{s\tau} - kx_{s\tau}$ cannot be stationary when $y, x \sim SI(1)$: our analysis shows that this statement is incorrect.

To date, no satisfactory practical method has been proposed for choosing between seasonal and periodic cointegration. Our analysis shows that this selection only has relevance when (in a bivariate context) both processes involved are seasonally integrated. The testing of the restrictions implied by periodic and seasnal cointegration is a topic for further research.

APPENDIX
Proofs

Corollary 1.1 Part (i) is trivially established by noting that y, x are conventional $I(1)$ variables and hence standard cointegration results imply that there is either a single cointegrating relationship between them or they are not cointegrated. In terms of the vectors Y_τ, X_τ, the common trend implied for each can be written in terms of any element $y_{s\tau}, x_{s\tau}$ and the single cointegrating relationship of Proposition 1 implies that $y_{s\tau} - kx_{s\tau}$ for some given s is stationary. It follows that $y_{s-i,\tau} = kx_{s-i,\tau}$ must also be stationary due to the stationarity of $y_{s\tau} - y_{s-i,\tau}$ and of $y_{s\tau} - y_{s-i,\tau}$ for any i. Thus, the cointegrating vector between any $y_{s\tau}, x_{j,\tau}$ $(s, j = 1, 2, 3, 4)$ is constant at $(1, -k)$ and hence full or partial periodic cointegration is impossible. Seasonal cointegration is ruled out by Definition 3 because this cointegration can occur only when the variables y, x are both $SI(1)$. Part (ii) is established by noting that stationarity of $y_{s\tau} - kx_{s\tau}$ and of $x_{s\tau} - x_{s-1,\tau}$ implies $y_{s\tau} - kx_{s-i,\tau}$ is stationary for any i.

With three linearly independent cointegrating relationships between $y_{s\tau}$ $(s = 1, 2, 3, 4)$ and another three cointegrating relationships between $x_{s\tau}$ $(s = 1, 2, 3, 4)$, the additional linearly independent cointegrating relationship $y_{s\tau} - kx_{s\tau}$ implies that A is of rank seven. The representation in part (iii) of the proposition is only one of many ways of specifying the seven linearly independent cointegrating relationships for the elements of Z_τ, but it is selected to emphasise the cointegration between each pair $y_{s\tau}$ and $x_{s\tau}$ $(s = 1, 2, 3, 4)$.

Corollary 1.2. Parts (i) and (ii) of this corollary are straightforward extensions of the corresponding parts of Corollary 1.1. From the $PI(1)$ properties of y and x, it follows that the final six elements of the vector defined by

$$K^{*\prime} Z_\tau = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -k_4^P \\ 1 & 0 & 0 & -\beta_1 & 0 & 0 & 0 & 0 \\ -\beta_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\alpha_1 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_3 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{1\tau} \\ y_{2\tau} \\ y_{3\tau} \\ y_{4\tau} \\ x_{1\tau} \\ x_{2\tau} \\ x_{3\tau} \\ x_{4\tau} \end{bmatrix} \tag{5.1}$$

are stationary. These final six rows of $K^{*\prime}$ define six linearly independent cointegraing vectors, where β_s are the periodic integration coefficients for y with $\beta_1\beta_2\beta_3\beta_4 = 1$, while α_s are the corresponding coefficients for x. Any stationary relationship which exists of the form $y_{s\tau} - k_s^P x_{s\tau}$ then defines a seventh linearly independent cointegrating vector. Without loss of generality, assume that this applies with $s = 4$ and here constitutes the first row of $K^{*\prime}$. Because of the stationarity of $y_{i\tau} - \beta_i y_{i-1,\tau}$ and of $x_{i\tau} - \alpha_i x_{i-1,\tau}$

$(i = 1,2,3,4)$ these seven relationships imply that $y_{s\tau} - k_s^P x_{s\tau}$ $(s = 1,2,3)$ is also stationary, where $k_1^P = \beta_1 k_4^P / \alpha_1, k_2^P = \beta_1 \beta_2 k_4^P / (\alpha_1 \alpha_2)$ and $k_3^P = \beta_1 \beta_2 \beta_3 k_4^P / (\alpha_1 \alpha_2 \alpha_3)$. When k_s^P are identical over $s = 1,2,3,4$ the cointegration is nonperiodic, otherwise it is periodic. Once again, seasonal cointegration is ruled out by definition and partial periodic cointegration cannot apply due to the cointegration which holds within the elements of Y_τ and within X_τ.

Part (iii) specifies the cointegrating matrix K'_{PP} in an analogous way to the corresponding matrix in Corollary 1.1 and follows from the stationarity of $y_{s\tau} - k_s^P x_{s\tau}$ $(s = 1,2,3,4)$. Part (iv) follows immediately from $k_1^P = \beta_1 k_4^P / \alpha_1, k_2^P = \beta_1 \beta_2 k_4^P / (\alpha_1 \alpha_2)$ and $k_3^P = \beta_1 \beta_2 \beta_3 k_4^P / (\alpha_1 \alpha_2 \alpha_3)$.

Corollary 1.3. This follows from the arguments used in establishing Corollary 1.2 with the $I(1)$ restrictions $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ applied for x.

Corollary 2.1. From (2.5), (2.6), (2.7) and (2.8) and taking $s = 4$, full seasonal cointegration immediately yields the cointegrating relationships given by the rows of K'_{SC} in (3.5). However, we also need to establish that these cointegrating relationships also apply for $s = 1,2,3$. It is straightforward to see that the cointegrating relationships of (2.5) and (2.6) lead to the first two rows of K'_{SC} irrespective of s, with (2.6) multiplied by -1 when s is odd. Similarly, (2.7) and (2.8) lead to the third and fourth rows respectively of K'_{SC} for $s = 2$, again after multiplying through by -1. Finally, for $s = 1$ or 3 the seasonal cointegrationg relationships (2.7) and (2.8) lead to these two rows of K'_{SC}, but now with the former leading to the fourth row of K'_{SC} and the latter to the third row. In all these cases, any $y_{j,\tau-1}$ or $x_{j,\tau-1}$ can be replaced by $y_{j\tau}$ or $x_{j\tau}$ respectively without affecting the cointegrating relationship, because $y_{j\tau} - y_{j,\tau-1}$ and $x_{j\tau} - x_{j,\tau-1}$ are stationary variables by nature of $y, x \sim SI(1)$.

For full periodic cointegration, the cointegrating relationships are of the form $y_{s\tau} - k_s^P x_{s\tau}$ with not all k_s^P equal over $s = 1, ..., 4$. This leads immediately to K'_{PC} in (3.6). The special case $k_1^P = k_2^P = k_3^P = k_4^P$ leads to nonperiodic cointegration between the $SI(1)$ processes. Full asynchronous periodic cointegration causes the coefficients k_s^P $(s = 1,2,3,4)$ to be shifted in the reght-hand 4×4 matrix of K'_{PC}, with the matrix 3.7) illustrating this for $i = 1$. Once again, the special case $k_1^P = k_2^P = k_3^P = k_4^P$ leads to the corresponding case of asynchronous nonperiodic cointegration.

Corollary 2.2. For the relationship between full periodic cointegration and full periodic cointegration in part (i), premultiply (3.6) by the transformation matrix T of (2.4). This yields

$$TK'_{PC} = \begin{bmatrix} 1 & 1 & 1 & 1 & -k_1^P & -k_2^P & -k_3^P & -k_4^P \\ -1 & 1 & -1 & 1 & k_1^P & -k_2^P & k_3^P & -k_4^P \\ -1 & 0 & 1 & 0 & k_1^P & 0 & -k_3^P & 0 \\ 0 & -1 & 0 & 1 & 0 & k_2^P & 0 & -k_4^P \end{bmatrix}. \tag{5.2}$$

which defines the cointegrating relationships of full seasonal cointegration only when the nonperiodic equality $k_1^P = k_2^P = k_3^P = k_4^P = k$ holds. Thus, in the nonperiodic case, the nonsingular linear transformation which produces the transformation of Y_τ used in seasonal cointegration does not produce the transformation of X_τ required for seasonal cointegration. Conversely, premultiplying the matrix of cointegrating vectors for full seasonal cointegration K'_{SC} by T^{-1} does yield the full periodic cointegration matrix either. To consider full asynchronous periodic cointegration and full periodic cointegration, it is sufficient to note that neither matrix can be obtained from the other through elementary row transformations. When full asynchronous periodic cointegration is considered in relation to full seasonal cointegration, analogous arguments apply to when the corresponding full periodic case is considered.

To establish part (ii), note that when $k_1^P = k_2^P = k_3^P = k_4^P = k$, (5.2) is

$$K'_{NPC} = \begin{bmatrix} 1 & 1 & 1 & 1 & -k & -k & -k & -k \\ -1 & 1 & -1 & 1 & k & -k & k & -k \\ -1 & 0 & 1 & 0 & k & 0 & -k & 0 \\ 0 & -1 & 0 & 1 & 0 & k & 0 & -k \end{bmatrix} \tag{5.3}$$

which is a special case of the seasonal cointegration cointegration matrix (3.5) with $k_1 = k_2 = k_R = k$ and $k_I = 0$. Since T is nonsingular the converse holds, so that this specific form of seasonal cointegration is equivalent to nonperiodic cointegration. Part (iii) can be established using the same approach.

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