# The natural rate hypothesis and the Inflation-output trade-offs in New Zealand and Turkey

**W A Razzak**
**Economics Department**
**Reserve Bank of New Zealand**
**P O Box 2498**
**Wellington, New Zealand**

**2000**

**Paper prepared for the Fourth International Conference in Economics Organised by the Economic Research Centre (ECR) of the Middle East Technical University in Ankara During the period 13-16 September 2000**

## Abstract

This paper tests the hypothesis that the higher the variance of aggregate demand shocks (or average prices) the less *favourable* the short-run trade-off between inflation and real output (deviations from trend) in New Zealand and Turkey. Results indicate that New Zealand's trade-off has improved significantly during the period of stable inflation, which can be a useful lesson for Turkey.

## 1. Introduction

Unlike the conventional Phillips curve (Phillips 1957), the natural rate hypothesis of Phelps (1967) and Friedman (1968) suggests that the trade-off between inflation and output (deviation from trend) does not arise from a *stable structure* of the economy. The immediate policy implication is that the monetary authority cannot choose a time path for inflation to keep real output above its potential level *permanently*. The natural rate hypothesis suggests that changes in average inflation do not increase average output, or in other words, the Phillips curve is vertical in the long run.

Lucas (1972) tests the natural rate theory. Lucas (1973) extends the theory by proposing that the higher the variance of average prices, the less favourable will be the observed short-run trade-off between inflation and output. A favourable *short-run* trade-off means that, ceteris paribus, short-run expansionary demand movements would increase real output with little or no inflationary consequences.

Recently there has been an increasing interest in inflation targeting as a policy regime (Bernanke and Mishkin, 1997). While New Zealand was the first to adopt such a monetary policy regime in 1989, Turkey has been embarked on a large reform programme, which includes a CPI inflation target of 25 percents, by the end of the year 2000, and 10-12 percent by the end of 2001.

This paper has a limited objective. It tests the natural rate's two propositions mentioned above using data from New Zealand and Turkey. It shows that stabilising the inflation rate in New Zealand resulted in an improved trade-off between inflation and output (deviations from trend). These results should be useful for Turkey.

In the next section the data are described. In section 3 a structural model, which embodies the natural rate hypothesis is spelt out. I estimate the model for both New Zealand and Turkey and discuss the results in section 4. Section 5 concludes.

## 2. The Data

Let $y_t$ denote the natural logarithm of real GDP, $x_t$ denote the logarithm of nominal GDP and $P_t$ denote the logarithm of the GDP deflator.

I plot growth rates of the three variables above for Turkey in figures 1a and 1b, and for New Zealand in figure 2.[1] The data are obtained from the IFS tapes. I do not use the CPI, but the difference between GDP deflator and the CPI is hopefully small. The data are quarterly from 1987 to 1998. Table 1 reports some descriptive statistics for Turkey and New Zealand.

For Turkey, visual inspection of the data reveals high inflation and high nominal income growth. The average growth rate of nominal income is about 60 percent, for the price level is approximately 55 percent, and they are highly correlated. Average growth rate of real output during the 10-year period is about 4.25 percent. While sample statistics are not sufficient to resolve issues like the trade-off between inflation and output, they suggest that average prices are independent of average real output. The variability of inflation measured by the standard deviation is approximately 8 percent and figure 1b shows that inflation fluctuates between 35 and 75 percent over the sample.

For New Zealand, during the disinflation period, inflation and nominal income growth seem to be highly correlated.[2] In 1991, New Zealand achieved its inflation target. From 1992 to 1998, inflation has been stable. It fluctuated around 1.6 percent with small variance. The correlation between inflation and nominal income growth has broken down. Real output growth increased from 0.3 percent during the period 1985-1991 to 3.3 during the period 1992-1998. Nominal income growth declined from 9 percent to about 5 percent during the same periods. Inflation declined from 8.7 to 1.6 percent. More importantly, the variability of nominal income growth measured by the standard deviation declined from 6.4 percent to 2.2 percent and the variability of inflation declined from 5.4 percent to less than 1 percent.

The standard deviation declined so sharply one need not statistically test the hypothesis that the variability of inflation has remained constant over time in New Zealand. Nor does one need to test the hypothesis that the standard deviation of nominal income growth and inflation in Turkey is equal to their New Zealand counterparts. Further, I do not report the statistics, but the ADF test for unit root indicates that the inflation rate in New Zealand is an I(0) variable for the period from 1992 onwards. Autocorrelation functions are also consistent with the ADF test. The Turkish inflation rate has no persistence whatsoever.

---

[1] $(\ln X_t - \ln X_{t-4}) * 100$.

[2] The New Zealand sample contains two policy regimes. These are the disinflation regime and the low-and-stable-regime from 1992 onward. The disinflation period started in mid 1980s, which is before the first Policy Target Agreement (1989).

## 3. The model

The model presented here is very simple. It is a model of the business cycle, which focuses on *variances* and not on the *persistence* (autocorrelation). The reason for choosing such a model is because my diagnosis of the data suggests that neither inflation nor nominal income growth is persistent in Turkey, though they are highly variable. In New Zealand, both the variability and the persistence have declined substantially during the period 1992 onwards. Real output growth, however, has become more persistent in New Zealand during the period of inflation targeting.

The objective of the model is to test Lucas' (1973) proposition that the higher the variance of average prices the less favourable the trade-off between inflation and output. There are three major assumptions underlying the model. Two of them are standard neo-classical assumptions. First, prices are assumed to be flexible. Sluggishness is not structural as in the Keynesian models, but arises from imperfect information. Second, output production is determined on the supply side of the economy. The last assumption is standard in macroeconomic models, namely that expectations are formed rationally (Muth, 1962).[3]

The model consists of a simple demand curve and a structural aggregate supply function (Lucas and Rapping, 1969). The aggregate demand curve is assumed to be a unit elastic very simple IS-LM type model.[4]

Let

$$y_t + P_t = x_t \qquad (1)$$

The variables are defined earlier. Let the shifter $x_t$ be an exogenous demand shock. For simplicity, assume that $\Delta x_t$ is a sequence of independent, normal variates with mean $\delta$ and a variance $\sigma_x^2$. The assumptions of exogeneity and independence are testable of course.

[3] Ball, Mankiw and Romer's (1988) sticky-price model includes the same proposition of Lucas's (1973) flexible price model. It predicts that lower average inflation makes the aggregate supply curve relatively flat so fluctuations in aggregate demand have large effects on output and are slowly reflected in prices. However, it offers a Keynesian interpretation of Lucas's result by saying that when inflation is high and variable, prices adjust more frequently.
[4] It is quite easy to include expected output, interest rate etc. in the demand curve. The model can be solved analytically, but estimation becomes impossible due to large number of parameters.

Also, they are not central to the theory of the natural rate, but maybe they are central to the formation of expectations (Lucas 73, p. 328, footnote 5).

To derive the aggregate supply function, output in each market $z$ is made up of two components, a secular or trend component (potential output), $y_{Pt}$, and a cyclical component $y_{ct}(z)$.

$$y_t(z) = y_{pt} + y_{ct}(z) \qquad (2)$$

Potential output (or trend output) reflects capital accumulation and population growth. It could be estimated using a linear trend

$$y_{pt} = \alpha + \beta t \qquad (3)$$

or using a filter such as the HP filter or the Baxter and King Band-Pass filter.

The cyclical component varies with perceived relative prices and it is persistent.[5]

$$y_{ct}(z) = \gamma[P_t(z) - E(P_t \mid \Omega_t(z))] + \phi y_{c,t-1}(z) \qquad (4)$$

The actual price level in market $z$ at time $t$ is $P_t(z)$. The *average* general price level is unobservable to suppliers. They form expectations about it, which are given by $E(P_t \mid \Omega_t(z))$, where $E$ is the expectations' operator, and it is conditioned on information available in market $z$.

Where do suppliers of goods and labour get their information? Suppliers are assumed to know *the history* of demand shifts, the history of changes in potential output and the history of cyclical movements. Historical information is not sufficient to determine the current general price level. However, it is sufficient to determine a *prior* distribution of the price level. It is assumed that this distribution is known with a given mean $\overline{P}_t$ and a constant variance $\sigma^2$.

The deviation of the actual price level in each market from the economy-wide average price level is distributed normally and independently of $P_t$. The distribution has a mean of zero and a variance $\tau^2$. Suppliers in each market use the information they have about this distribution to compute

[5] There is no general agreement on the dynamics and on ***why*** cycles are persistent.

the distribution of $P_t$ conditional on the price of their own goods and the mean of the history of average prices $\overline{P}_t$.

Averaging over markets gives the Lucas aggregate supply function, which relates cyclical real output to relative prices.

$$y_t = y_{pt} + \theta\gamma[P_t - \overline{P}_t] + \phi[y_{t-1} - y_{p,t-1}]. \qquad (5)$$

The cyclical component of real output varies with changes in *relative prices*, where the parameter $\gamma$ captures the responsiveness of the cyclical component of real output to changes in *relative prices*.

The parameter $\theta$ depends on $\tau^2$ (the variance of the distribution of the deviation of the price in a local market from the average price level) and $\sigma^2$ (the variance of the distribution of the average price level) or in the other words, the relative variances of shocks. It implies that the suppliers' main task is to distinguish whether demand shifts constitute an increase in aggregate demand or an increase in the demand in their specific market (signal extraction).

To make a real gain, the central bank can surprise suppliers by expansionary monetary policy, which shifts the aggregate demand curve up. Typical suppliers who observe the actual price level to be higher than their expected price may increase their production in the short run and the central bank would achieve its goal of higher real output. Suppliers will realise that they have confused aggregate demand shifts with local demand shifts. When they realise this they cut down production and output returns to equilibrium. The reverse is also true when the central bank pursues a surprisingly tight monetary policy. The model, however, allows this to happen for one period only. The assumption is that rational suppliers cannot be fooled twice.

The model is solved for the equilibrium price and output equations by combining both the aggregate demand and the aggregate supply.

$$y_t = -\frac{\theta\gamma\delta}{1+\theta\gamma} + \phi\beta + \frac{\theta\gamma}{1+\theta\gamma}\Delta x_t + \phi y_{t-1} + (1-\phi)y_{p,t} \qquad (6)$$

$$P_t = \frac{\theta\gamma\delta}{1+\theta\gamma} - \phi\beta + \frac{1}{1+\theta\gamma}x_t + \frac{\theta\gamma}{1+\theta\gamma}x_{t-1} - \phi y_{t-1} - (1-\phi)y_{p,t} \qquad (7)$$

Nominal income is used as a proxy for exogenous demand shocks. Equation (6) describes real output as a positive function of nominal output growth $\Delta x_t$, lagged real output $y_{t-1}$ and potential output $y_{p,t}$. The assumption that $\Delta x_t$ is a sequence of independent normal variates is examined.[6] The parameters and their signs represent theoretical restrictions.

Equation (7) describes the log of the price level $P_t$ as a positive function of nominal output $x_t$ and its lagged value; a decreasing function of lagged real output $y_{t-1}$, and potential output $y_{p,t}$.[7] Thus, for example, when the AS curve shifts to the right real output increases and the price level falls.

In terms of deviations of output from potential (output gap) and inflation, equation (6) and (7) become:

$$y_{ct} = -\delta\eta + \eta\,\Delta x_t + \phi\, y_{c,t-1} + u_{1t} \tag{8}$$

$$\Delta P_t = -\beta + (1-\eta)\,\Delta\, x_t + \eta\,\Delta x_{t-1} - \phi\,\Delta\, y_{c,t-1} + u_{2t} \tag{9}$$

Derivation of (8) and (9) is found in the appendix. Recall that $\beta$ in (9) is the trend of real output. The conditional variance of $P_t$ will be a constant equal to:

$$\sigma^2 = \frac{1}{(1+\gamma\theta)^2\sigma_x^2}\,. \tag{10}$$

The model asserts the existence of the natural rate hypothesis. The parameter $\delta$ is the average rate of demand expansion. It has a negative sign. It means that changes in the average rate of nominal output growth have no effect on average real output in the long run. For example, in the long run, $y_{c,t}$ is zero, so $\delta = \Delta x_t$. The natural rate paradigm implies that there exists no unique inflation rate such that the central bank can choose to keep real output higher than its normal level (i.e. potential) *permanently*.

---

[6] Razzak (1999) tests nominal income growth for white noise using the Kolmogrov-Smirnov test. He shows that it is a white noise process for the period of stable inflation in New Zealand. For Turkey, the test value is 0.1628, which is less than the 95% critical value 0.2835. Therefore, nominal income growth is a white noise process.

[7] Also, nominal GDP may not be a very good proxy for demand because macro data have serious measurements' problems and limited amount of information.

Combining $\eta = \frac{\theta\gamma}{1+\theta\gamma}$ and $\theta = \sigma^2 + \tau^2$ we arrive at $\eta = \frac{\tau^2\gamma}{\sigma^2 + \tau^2(1+\gamma)}$.
Substituting for $\sigma^2$ from (5) gives:

$$\eta = \frac{\tau^2\gamma}{(1-\eta)^2\sigma_x^2 + \tau^2(1+\gamma)} \qquad (11)$$

The model suggests that as the variance of aggregate demand shift approaches zero, the value of $\eta$ approaches $\frac{\gamma}{1+\gamma}$ and tends monotonically to zero as the variance of aggregate demand shift approaches infinity. Thus, under a regime of stable inflation and a central bank that is not trading on the Phillips curve, the value of the estimated $\eta$ is larger than that under a regime of high inflation and a central bank that exploits the Phillips curve.

## 4. Estimation

The variables were defined earlier, however, the output gap $y_{ct}$ is computed using the HP filter.[8] I provide estimates of the system (8) and (9) for New Zealand and Turkey.[9] The Generalised Method of Moments (GMM) is used to estimate the system. Table 2 reports the estimates.[10]

New Zealand's $\eta$ is 0.05 during the period of disinflation, where inflation was high but falling. For Turkey, $\eta$ is 0.09, which is pretty close to the New Zealand's estimate. A small value of $\eta$ indicates unfavourable trade-offs between inflation and output, but New Zealand was disinflating at a very high rate, while Turkey only started disinflation recently. The trade-off is unfavourable in the sense that the increase in nominal income increases inflation by more than it increases real output (relative to trend).

Just like the model would predict, the size of $\eta$ increased to 0.53 in New Zealand during the period of stable inflation (1992 onwards). It maybe interpreted as the Reserve Bank of New Zealand pursued credible inflation policies and did not attempt to exploit the trade-off between inflation and output by surprising people and purposefully shifting the aggregate demand curve. Stable demand management leads to a lower

[8] It is probably inappropriate to use a smoothing parameter 1600 for both New Zealand and Turkey, but investigating the appropriate parameter is beyond the scope of this paper.
[9] I borrowed the results for New Zealand from Razzak (1999), where the sample ends in 1998:1. I have updated the estimates, but have not reported them here. Results are available upon request.
[10] Razzak (1999) provides time series and pooled time-series and cross-section estimates of equations 8 and 9 individually and as a system.

variability of the inflation rate, which leads to a larger $\eta$ and to a favourable trade-off.

The coefficient $\delta$ reflects the level of inflation associated with a zero output gap. For New Zealand, when the output gap is zero, inflation is slightly above one percent. For Turkey, it is about 13 percent. This is the estimate of the long-run inflation rate that corresponds to the natural rate of output. The coefficients have the expected signs.

The size of the persistence coefficient $\phi$ is the same in New Zealand's first sub-sample and Turkey. The coefficient becomes very large (doubled) in magnitude in New Zealand during the period of stable inflation. So why does output become more persistent? There is no clear answer to this question and it depends on the model of the economy. I speculate that the reason behind the increase in the size of $\phi$ maybe related to the effectiveness of policy in responding to demand shocks. If output is decomposed into transitory and permanent components, the transitory component is offset by policy while the permanent component is left to dominate output fluctuations.

I fit a trend into the inflation equation for New Zealand and found it to be insignificant. The trend in Turkey is a little significant and has the expected sign.

## 5. Conclusions

I have tested the natural rate hypothesis in New Zealand over periods of disinflation and periods of the low and stable inflation during the 1990s. I also compared the results with those of Turkey, a high and a variable inflation country.

The conventional Phillips curve story suggests that the trade-off between inflation and output arise from a stable structure of the economy. In this paper I use Lucas' (1973) model, which assumes that the trade-off between inflation and output arises because suppliers confuse general prices movements with relative price changes. The story is based on two important views: First is the Phelps-Friedman view that a higher average inflation does not increase average output in the long run. This is quite clear in the sample data. Second is Lucas's proposition that the higher the variance of average prices, the less favourable will be the trade-off. This is also evident in both the New Zealand and the Turkish data.

## References

Ball, L., G. Mankiw and D. Romer, (1988), "The New Keynesian Economics and the Output-Inflation Trade off," *Brooking Papers on Economic Activity*, 1-65.

Bernanke, B. and F. Mishkin, (1997), "Inflation Targeting: A New Framework for Monetary policy" *Journal of Economic Perspectives* 11, no.2 (spring), 97-116.

Friedman, M., (1968)"The Role of Monetary Policy," *American Economic Review*, 1-17.

Gallant, R. (1987), Nonlinear Statistical Models, New York, Wiley.

Gallant, R. and D. W. Jorgenson, (1979), "Statistical Inference in an Implicit Nonlinear, Simultaneous Equation Model in the Context of Maximum Likelihood Estimation," *Journal of Econometrics* 11, 275-302.

Hansen, L. P., (1982), "Large Sample Properties of Generalised Method of Moments Estimators," *Econometrica* 56, 1029-1054.

Hodrick, R., and E. C. Prescott, (1997), "Post war business cycles: An empirical Investigation," *Journal of Money Credit and Banking*, 1-16.

Lucas, R. JR., (1973), "Some International Evidence on Output-Inflation Tradeoffs," *American Economic Review* Vol. 63, No. 3, 326-334.

Lucas, R. JR., (1972), "Econometric Testing of the Natural-Rate Hypothesis," The Econometrics of Price Determination, Otto Eckstein, ed., *Board of Governors of the Federal Reserve System*, 56-59.

Lucas, R. JR. and L. A. Rapping (1969), "Real Wages, Employment and the Price Level," *Journal of Political Economy* 77, 721-754.

Muth, J., (1961), "Rational Expectations and the Theory of Price Movements," *Econometrica* 29, 315-335.

Phelps, E., (1969), Microeconomics of Inflation and Employment, Introductory chapter, in E. Phelps *et al*, New York.

Razzak, W. A., (1999), "Are Inflation Targeting Regimes Credible? Econometric Evidence. Unpublished Manuscript presented at the Money, Macro and Finance Annual Meeting in London, September 6-8, 2000.

Table 1. Descriptive Statistics

| | $\pi$ | | $y$ | | $x$ | |
|---|---|---|---|---|---|---|
| | Mean | STD | Mean | STD | Mean | STD |
| New Zealand | | | | | | |
| *(1985:2-1991:4)* | 8.77 | 5.45 | 0.32 | 1.49 | 9.09 | 6.39 |
| *(1992:1-1998:1)* | 1.59 | 0.80 | 3.35 | 1.95 | 4.94 | 2.23 |
| | | | | | | |
| Turkey | | | | | | |
| *(1988:1-1998:4)* | 54.98 | 7.91 | 4.24 | 5.28 | 59.36 | 7.98 |

$\pi$ is defined as $(\ln P_t - \ln P_{t-4})*100$, where $P_t$ is GDP deflator.

$y$ is defined as $(\ln y_t - \ln y_{t-4})*100$, where $y_t$ is real GDP.

$x$ is defined as $(\ln x_t - \ln x_{t-4})*100$, where $x_t$ is nominal GDP.

Table 2
GMM Estimates of the model

| Country | $\eta$ | $\phi$ | $\delta$ | $\beta$ | $\chi^2_3$ | *wald* $P > \chi^2_1$ $H_0 : \eta_1 = \eta_2$ |
|---|---|---|---|---|---|---|
| New Zealand | | | | | | |
| 85:2-91:4 | 0.05 | 0.44 | -1.25 | NA | 4.48 | |
| | *(0.1612)* | *(0.0001)* | *(0.5718)* | | | |
| 92:1-98:1 | 0.53 | 0.85 | -1.11 | NA | 5.84 | |
| | *(0.0001)* | *(0.0001)* | *(0.0001)* | | | ***0.0001*** |
| Turkey | | | | | | |
| 88:1 – 98:4 | 0.09 | 0.34 | -13.5 | -0.98 | 8.19 | NA |
| | *(0.0484)* | *(0.0001)* | *(0.0018)* | *(0.0310)* | | |

- The endogenous variables are the output gap and inflation $(y_{ct}, \Delta P_t)$. The rest $(\Delta x_t, \Delta x_{t-1}, y_{c,t-1}, \Delta y_{c,t-1})$ and time trend are predetermined and used as (unique) instruments.
- The Hansen (1982) test statistics for over-identifying restrictions of the instruments are not reported, but they are all insignificant. The P values of $t$ the statistics are in parentheses.
- $\chi^2$ test statistics for the restricted model (Gallant and Jorgenson, 1979). The 5% and 10% critical values with three degrees of freedom are 7.81 and 11.34 respectively.

Figure 1a: Turkey growth rates of real, nominal GDP and P

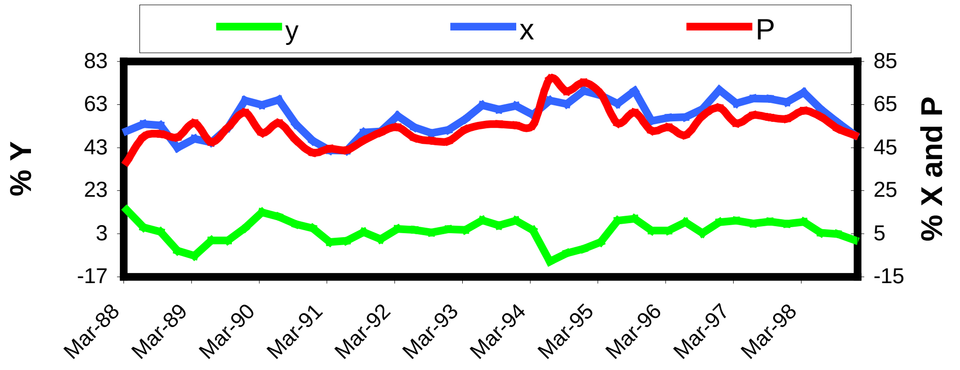



Figure 1b: Turkish Inflation

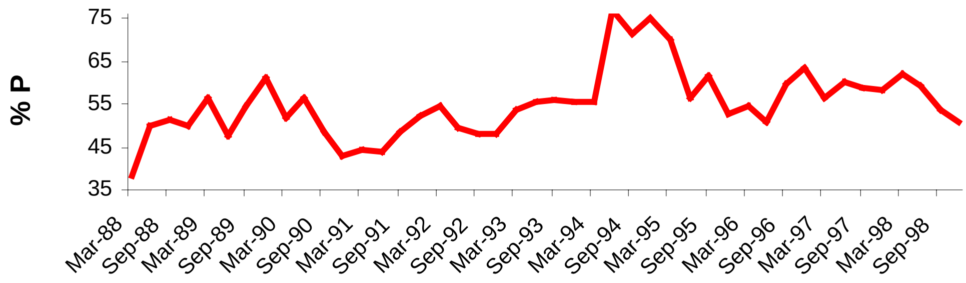
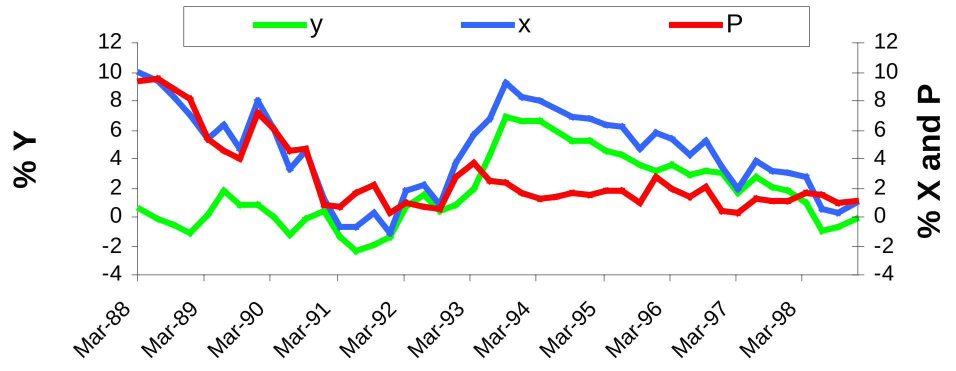



Figure 2:New Zealand growt rates of real, Nominal GDP and the price level



**Appendix**

Let $z$ index markets. And output is decomposed into a normal (secular) component $y_{nt}$ and a cyclical one, which varies from one market to another $y_{ct}(z)$. All variables are in logs.

$$y_t(z) = y_{nt} + y_{ct}(z) \quad (1)$$

The trend reflects capital accumulation and population growth change.

$y_{nt} = \alpha + \beta t$ (This could be modelled differently of course). (2)

The cyclical component varies with perceived relative prices and with its own lagged value.

$$y_{ct}(z) = \gamma[P_t(z) - E(P_t \mid I_t(z)] + \phi y_{c,t-1}(z) \quad (3)$$

The actual price in market $z$ is $P(z)$ at time $t$ and $E_t(P_t \mid I_t(z)$ is the mean current general price level, conditioned on information available in $z$ at time $t$.

It is assumed that actual price deviates from the geometric economy-wide price level by an amount that is distributed independently of $P_t$. The percentage deviation of the price level in $z$ from average $P_t$ is denoted $z$ (so that markets are indexed by the their price deviations from average) where $z_t$ is normally distributed, independent of $P_t$, with mean zero and variance $\tau^2$ ($Cov(P,z) = 0$). The observed log price in $z$, $P_t(z)$ is the sum of independent normal variates.[11]

$$P_t(z) = P_t + z \quad (4)$$

Rational forecasts

$$E(P_t(z) \mid I_t) = E(P_t \mid I_t) + E(z \mid I_t) = E(P_t \mid I_t) = \overline{P}_t \quad (5)$$

$$Var(P_t(z)) = Var(P_t) + Var(z) + 2Cov(P_t, z) = \sigma_P^2 + \tau^2 \quad (6)$$

$$Cov(P_t(z), P_t) = E[P_t(z) - E(P_t(z) \mid I_t)][P_t - E(P_t \mid I_t)] \quad (7)$$

$$= E[(P_t(z) - \overline{P}_t) - (P_t - \overline{P}_t)] \quad (8)$$

$$= E[(P_t + z - \overline{P}_t)(P_t - \overline{P}_t)] \quad (9)$$
$$= E(P_t - \overline{P}_t)^2 + E[(z\ )(P_t - \overline{P}_t)] \quad (10)$$

---

[11] $z$ is not necessarily a constant.

$= \sigma_P^2$ because $z$ and $P_t$ are independent and $\overline{P}_t = \overline{P} =$ a constant.

$$\therefore E(P_t \mid I_t(z)) = \overline{P}_t + \frac{\sigma_P^2}{\sigma_P^2 + \tau^2}(P_t(z) - \overline{P}_t) \qquad (11)$$

$$= \frac{\sigma_P^2}{\sigma_P^2 + \tau^2} P_t(z) + \overline{P}_t(1 - \frac{\sigma_P^2}{\sigma_P^2 + \tau^2}) \qquad (12)$$

$$= (1 - \frac{\sigma_P^2}{\sigma_P^2 + \tau^2})P_t(z) + (\frac{\sigma_P^2}{\sigma_P^2 + \tau})\overline{P}_t \qquad (13)$$

(1) $= (1-\theta)P_t(z) + \theta\overline{P}_t$, where $\theta = \frac{\sigma_P^2}{\sigma_P^2 + \tau^2}$ (14)

Substitute (3) and into (1)

$$y_t(z) = y_{nt} + \gamma[P_t(z) - E(P_t \mid I_t(z)] + \phi y_{c,t-1}(z) \qquad (15)$$

And using (5) we get

$$y_t(z) = y_{nt} + \gamma P_t(z) - \gamma[(1-\theta)P_t(z) + \theta\overline{P}_t] + \phi y_{c,t-1}(z) \qquad (16)$$

$$y_t(z) = y_{nt} + \theta\gamma[P_t(z) - \overline{P}_t] + \phi y_{c,t-1}(z) \qquad (17)$$

From equation (1) we get

$$y_{c,t-1}(z) = y_{t-1}(z) - y_{n,t-1} \qquad (18)$$

$$\therefore y_t(z) = y_{nt} + \theta\gamma[P_t(z) - \overline{P}_t] + \phi[y_{t-1}(z) - y_{n,t-1}] \qquad (19)$$

Aggregating over all markets, we get the Lucas aggregate supply curve

$$y_t = y_{n,t} + \theta\gamma[P_t - \overline{P}_t] + \phi[y_{t-1} - y_{n,t-1}] \qquad (20)$$

The aggregate demand is a simple IS-LM unit elastic

$$y_t + P_t = x_t \qquad (21)$$

Equate the aggregate demand to the aggregate supply

$$(x_t - P_t) = y_{nt} + \theta\gamma[P_t - \overline{P}_t] + \phi[y_{t-1} - y_{n,t-1}] \qquad (22)$$

$$x_t - y_{nt} + \theta\gamma\overline{P}_t - \phi[y_{t-1} - y_{n,t-1}] = P_t + \theta\gamma P_t \qquad (23)$$

$$P_t(1+\theta\gamma) = x_t - y_{nt} + \theta\gamma\overline{P}_t - \phi[y_{t-1} - y_{n,t-1}] \qquad (24)$$

$$\overline{P}_t = E(P_t \mid I_t) \qquad (25)$$

From the demand equation 21

$$P_t = x_t - y_t \Rightarrow E(P_t \mid I_t) = E(x_t \mid I_t) - E(y_t \mid I_t) \Rightarrow \overline{P}_t = E(x_t \mid I_t) - E(y_t \mid I_t) \quad (26)$$

$\Delta x_t$ is distributed normal with a mean $\delta$ and a variance $\sigma_x^2$, thus,

$$E(x_t - x_{t-1} \mid I_t) = \delta \quad (27)$$

$$E(x_t \mid I_t) - E(x_{t-1} \mid I_t) = \delta \quad (28)$$

$$E(x_t \mid I_t) = \delta + x_{t-1} \quad (29)$$

From the aggregate supply equation

$$y_t = y_{n,t} + \theta\gamma[P_t - \overline{P}_t] + \phi[y_{t-1} - y_{n,t-1}] \quad (30)$$

and

$$\begin{aligned} E(y_{nt} \mid I_t) &= y_{nt} \\ E(P_t \mid I_t) &= \overline{P}_t \\ E(y_{t-1} \mid I_t) &= y_{t-1} \end{aligned} \quad (31)$$

$$\therefore E(y_t \mid I_t) = y_{nt} + \phi[y_{t-1} - y_{n,t-1}] \quad (32)$$

Hence

$$P_t(1+\theta\gamma) = x_t - y_{nt} + \theta\gamma(\delta + x_{t-1} - y_{nt} - \phi[y_{t-1} - y_{n,t-1}]) - \phi(y_{t-1} - y_{n,t-1}) \quad (33)$$

$$P_t(1+\theta\gamma) = x_t - y_{nt} + \theta\gamma\delta + \theta\gamma x_{t-1} - \theta\gamma y_{nt} - \theta\gamma\phi(y_{t-1} - y_{n,t-1}) - \phi(y_{t-1} - y_{n,t-1}) \quad (34)$$

$$P_t(1+\theta\gamma) = x_t - (1+\theta\gamma)y_{nt} + \theta\gamma\delta + \theta\gamma x_{t-1} - \phi(1+\theta\gamma)(y_{t-1} - y_{n,t-1}) \quad (35)$$

And

$$P_t = \frac{\theta\gamma\delta}{1+\theta\gamma} + \frac{1}{1+\theta\gamma}x_t + \frac{\theta\gamma}{1+\theta\gamma}x_{t-1} - y_{nt} - \phi(y_{t-1} - y_{n,t-1}) \quad (36)$$

$$P_t = \frac{\theta\gamma\delta}{1+\theta\gamma} + \frac{1}{1+\theta\gamma}x_t + \frac{\theta\gamma}{1+\theta\gamma}x_{t-1} - y_{nt} - \phi y_{t-1} + \phi y_{n,t-1} \quad (37)$$

But

$$y_{nt} = \alpha + \beta t \quad (38)$$

And

$$y_{n,t-1} = \alpha + \beta(t-1) = \alpha + \beta t - \beta = y_{nt} - \beta \quad (39)$$

$$\therefore \phi y_{n,t-1} - y_{nt} = \phi y_{nt} - \phi\beta - y_{nt} = -(1-\phi)y_{nt} - \phi\beta \quad (40)$$

Which gives us

$$P_t = \frac{\theta\gamma\delta}{1+\theta\gamma} + \frac{1}{1+\theta\gamma}x_t + \frac{\theta\gamma}{1+\theta\gamma}x_{t-1} - \phi\beta - \phi y_{t-1} - (1-\phi)y_{n,t} \quad (41)$$

Substitute the above solution into the aggregate demand equation (21) and solve for $y_t$.

$$y_t = x_t - \frac{\theta\gamma\delta}{1+\theta\gamma} + \phi\beta - \frac{1}{1+\theta\gamma}x_t - \frac{\theta\gamma}{1+\theta\gamma}x_{t-1} + \phi y_{t-1} + (1-\phi)y_{nt} \quad (42)$$

$$y_t = (1 - \frac{1}{1+\theta\gamma})x_t - \frac{\theta\gamma}{1+\theta\gamma}x_{t-1} + \phi\beta - \frac{\theta\gamma\delta}{1+\theta\gamma} + \phi y_{t-1} + (1-\phi)y_{nt} \quad (43)$$

$$y_t = (\frac{\theta\gamma}{1+\theta\gamma})x_t - \frac{\theta\gamma}{1+\theta\gamma}x_{t-1} + \phi\beta - \frac{\theta\gamma\delta}{1+\theta\gamma} + \phi y_{t-1} + (1-\phi)y_{nt} \quad (44)$$

$$y_t = -\frac{\theta\gamma\delta}{1+\theta\gamma} + \phi\beta + \frac{\theta\gamma}{1+\theta\gamma}\Delta x_t + \phi y_{t-1} + (1-\phi)y_{nt} \quad (45)$$

Now let $\frac{\theta\gamma}{1+\theta\gamma} = \eta$ (46)

$$y_t = -\eta\delta + \eta\Delta x_t + \phi\beta + \phi y_{t-1} + (1-\phi)y_{nt} \quad (47)$$

And from equation (1)

$$y_{ct} = y_t - y_{nt} \quad (48)$$

$$y_{ct} = -\eta\delta + \eta\Delta x_t + \phi\beta + \phi y_{t-1} + (1-\phi)y_{nt} - y_{nt} \quad (49)$$

$$y_{ct} = -\eta\delta + \eta\Delta x_t + \phi\beta + \phi y_{t-1} \phi y_{nt} \quad (50)$$

Since $y_{n,t-1} = y_{nt} - \beta$ and $y_{nt} = y_{n,t-1} + \beta$ (51)

Therefore,

$$y_{ct} = -\eta\delta + \eta\Delta x_t + \phi\beta + \phi y_{t-1} - \phi(y_{n,t-1} + \beta) \quad (52)$$

$$y_{ct} = -\eta\delta + \eta\Delta x_t + \phi(y_{t-1} - y_{n,t-1}) \quad (53)$$

$$\boxed{y_{ct} = -\eta\delta + \eta\Delta x_t + \phi y_{c,t-1}} \quad (54)$$

The lagged value of $P_t$ from (41)

$$P_{t-1} = \frac{\theta\gamma\delta}{1+\theta\gamma} - \phi\beta + \frac{1}{1+\theta\gamma}x_{t-1} + \frac{\theta\gamma}{1+\theta\gamma}x_{t-2} - \phi y_{t-2} - (1-\phi)y_{n,t-1} \quad (55)$$

Subtracting (55) from (41) yields

$$\Delta P_t = (1-\eta)\Delta x_t + \eta\Delta x_{t-1} - \phi(y_{t-1} - y_{t-2}) - (1-\phi)(y_{nt} - y_{n,t-1}) \qquad (56)$$

Again, recall that $y_{nt} = \alpha + \beta t$, $y_{n,t-1} = \alpha + \beta(t-1) = \alpha + \beta t - \beta$ (57)

So $y_{nt} - y_{n,t-1} = \beta$ and $y_{n,t-2} = \alpha + \beta(t-2) = \alpha + \beta t - 2\beta$, (58)

And $y_t = y_{nt} - y_{ct}$, $y_{t-1} = y_{n,t-1} - y_{c,t-1}$ and $y_{t-2} = y_{n,t-2} - y_{c,t-2}$ (59)

$$\therefore y_{t-1} - y_{t-2} = \beta + y_{c,t-1} - y_{c,t-2} = \beta + \Delta y_{c,t-1} \qquad (60)$$

Now substitute (60) in (56)

$$\Delta P_t = (1-\eta)\Delta x_t + \eta\Delta x_{t-1} - \phi(\beta + \Delta y_{c,t-1}) - (1-\phi)\beta \qquad (61)$$

$$\boxed{\Delta P_t = -\beta + (1-\eta)\Delta x_t + \eta\Delta x_{t-1} - \phi y_{c,t-1}} \qquad (62)$$